Normal subgroups

Let G be a group. A subgroup $H \leq G$ is <u>normal</u> if, $\forall g \in G$,

$$gHg^{-1} = \{ghg^{-1} : h \in H\} \subseteq H. \qquad \text{Notation: } H \trianglelefteq G.$$

ghg^{-1}: conjugate of h by g

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Motivation: Normal subgroups are precisely what are needed to define "quotient groups" G/H, which are analogues of $\mathbb{Z}/n\mathbb{Z}$, for other groups.

Exs:

1a) Suppose G is any group and let $H = \{e\}$.

Then $\forall g \in G$, $geg^{-1} = gg^{-1}e = e \in H$.

Therefore $\{e\} \trianglelefteq G$.

1b) Suppose G is any group and let $H = G$.

Then $\forall g \in G, h \in H$, $ghg^{-1} \in G = H$.

Therefore $G \trianglelefteq G$.

Note: Groups G whose only normal subgroups are $\{e\}$ and G are called <u>simple groups.</u> Some examples of simple groups are:

C_p for p prime (the only Abelian simple groups)

A_5 (the smallest non-Abelian simple group)

A_n for $n \geq 5$

2) If G is Abelian then every subgroup $H \leq G$ is normal.

$\forall h \in H,\ g \in G, \quad ghg^{-1} \underset{\text{G is Abelian}}{=} (gg^{-1})h = h \in H$

Therefore $gHg^{-1} \subseteq H$.

3a) $G = S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), \underbrace{(1\ 2\ 3)}_{=(13)(12)}, \underbrace{(1\ 3\ 2)}_{=(12)(13)}\}$

$H = \langle (1\ 2\ 3) \rangle = \{e, (1\ 2\ 3), \underbrace{(1\ 3\ 2)}_{=(1\ 2\ 3)^2}\}$

Note: • $H = A_3$ (subgroup of S_3 consisting of all even perms.)

• $\forall g \in G,\ h \in H,$

h is even $\Rightarrow ghg^{-1}$ is even $\Rightarrow ghg^{-1} \in H$.

Conclusion: $H \trianglelefteq G$.

3b) $G = S_3, \quad H = \langle (1\ 2) \rangle = \{e, (1\ 2)\}$

$(2\ 3)(1\ 2)\underbrace{(2\ 3)^{-1}}_{=(2\ 3)} = (1\ 3) \notin H$

Conclusion: $H \ntrianglelefteq G$.

Equivalent characterizations of normal subgroups

Def: $\forall g, g' \in G$, $\forall S \subseteq G$, define

$gS = \{gs : s \in S\}$, $Sg = \{sg : s \in S\}$, and $gSg' = \{gsg' : s \in S\}$.

Theorem: Suppose G is a group and H is a subgroup of G.

The following statements are equivalent:

i) $\forall g \in G$, $gHg^{-1} \subseteq H$. $(H \trianglelefteq G)$

ii) $\forall g \in G$, $gHg^{-1} = H$.

left coset of H by g

iii) $\forall g \in G$, $gH = Hg$.

right coset of H by g

Pf: i) $\Rightarrow$ ii): Suppose i) holds. We need to show that $\forall g \in G$, $H \subseteq gHg^{-1}$.

Note that $\forall g \in G$, we have $g^{-1} \in G$ so $g^{-1}H(g^{-1})^{-1} = g^{-1}Hg \subseteq H$.

Therefore, $\forall h \in H$, $g^{-1}hg \in H$

$\Rightarrow g^{-1}hg = h'$, for some $h' \in H$

$\Rightarrow h = gh'g^{-1} \Rightarrow h \in gHg^{-1}$.

Therefore $H \subseteq gHg^{-1}$, so $H = gHg^{-1}$.

(... $\Rightarrow$ next page)

ii) $\Rightarrow$ iii) : Suppose ii) holds. Then:

- $\forall g \in G$, $h \in H$, $ghg^{-1} = h'$ for some $h' \in H$

  $\Rightarrow gh = h'g \Rightarrow gh \in Hg$.

  Therefore $gH \subseteq Hg$.

- $\forall g \in G$, $h \in H$, $\exists h' \in H$ s.t. $gh'g^{-1} = h$

  $\Rightarrow hg = gh' \Rightarrow hg \in gH$.

  Therefore $Hg \subseteq gH$, so $gH = Hg$.

iii) $\Rightarrow$ i) is similar. $\blacksquare$

Other important facts

1) Suppose G is a group and H is a subgroup of G. Then, $\forall g \in G$, the set gHg^{-1} is also a subgroup of G, and the map

$$\phi_g : H \to gHg^{-1} \quad \text{defined by} \quad \phi_g(h) = ghg^{-1}$$

is an isomorphism.

Pf: Suppose $g \in G$, and consider the map $\tilde{\phi}_g : H \to G$, $\tilde{\phi}_g(h) = ghg^{-1}$.

For any $h, h' \in H$,

$$\tilde{\phi}_g(hh') = ghh'g^{-1} = gh(g^{-1}g)h'g^{-1} = (ghg^{-1})(gh'g^{-1}) = \tilde{\phi}_g(h)\tilde{\phi}_g(h').$$

Therefore $\tilde{\phi}_g$ is a homom. $\Rightarrow gHg^{-1} = \tilde{\phi}_g(H) \leq G$. (property 2 of homoms.)

It follows that $\phi_g : H \to gHg^{-1}$, $\phi_g(h) = ghg^{-1}$ is a surjective homom.

Finally, if $h, h' \in H$ and $\phi_g(h) = \phi_g(h')$ then $ghg^{-1} = gh'g^{-1} \Rightarrow h = h'$. Therefore ϕ_g is also injective.

Conclusion: ϕ_g is an isomorphism. $\blacksquare$

Ex:

4) $G = D_8 = \langle r, s \mid r^4 = s^2 = e,\ rs = sr^{-1} \rangle = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$

$H = \langle r \rangle = \{e, r, r^2, r^3\}$

Note: $\forall g \in G$, $gHg^{-1} \leq G$, and $gHg^{-1} \cong H \cong C_4$.

Therefore $gHg^{-1} = \langle x \rangle$, for some $x \in G$ with $|x| = 4$.

Then $x = r$ or r^3 $\Rightarrow gHg^{-1} = H$.

only elems of order 4 in D_8.

Conclusion: $H \trianglelefteq G$.

General comment: Any time a group G contains a subgroup H which is not isomorphic to any other subgroup of G, we have $gHg^{-1} \cong H \Rightarrow gHg^{-1} = H \Rightarrow H \trianglelefteq G$.

2) If G and K are groups and $\phi: G \to K$ is a homomorphism, then $\ker(\phi) \trianglelefteq G$.

$= \{g \in G : \phi(g) = e_K\}$

Pf: We already proved that $\ker(\phi) \leq G$.

$\forall g \in G$, $h \in \ker(\phi)$,

$$\phi(ghg^{-1}) \underset{\phi \text{ is a hom.}}{=} \phi(g)\overset{=e}{\phi(h)}\phi(g)^{-1} = \phi(g)\phi(g)^{-1} = e.$$

Therefore $ghg^{-1} \in \ker(\phi)$. ∎

Ex: 5) The map $\phi: GL_2(\mathbb{R}) \to \mathbb{R}\setminus\{0\}$, $\phi(A) = \det(A)$, is a homomorphism with (see "Basic properties of homomorphisms")

$$\ker(\phi) = \{A \in GL_2(\mathbb{R}) : \det(A) = 1\} = SL_2(\mathbb{R})$$

Therefore $SL_2(\mathbb{R}) \trianglelefteq GL_2(\mathbb{R})$.

3) If $H \leq G$, $G = \langle S \rangle$, and $H = \langle T \rangle$, then:

i) $H \trianglelefteq G$ if and only if, $\forall s \in S$, $t \in T$,

$sts^{-1} \in H$, $s^{-1}ts \in H$, $st^{-1}s^{-1} \in H$, and $s^{-1}t^{-1}s \in H$. (*)

ii) If $|G| < \infty$ then $H \trianglelefteq G$ if and only if, $\forall s \in S$, $t \in T$,

$sts^{-1} \in H$.

Pf of i): The implication $\Rightarrow$ follows from the def. of "normal subgroup".

To prove $\Leftarrow$, suppose that $\forall s \in S, t \in T$, condition (*) holds.

Step 1: Show that, $\forall s \in S$, $h \in H$, $shs^{-1} \in H$, $s^{-1}hs \in H$. ✓

Since $H = \langle T \rangle$, $\forall h \in H$, $\exists t_1, \ldots, t_n \in T$, $u_1, \ldots, u_n \in \{\pm 1\}$ s.t. $h = t_1^{u_1} t_2^{u_2} \cdots t_n^{u_n}$.

Then, $\forall s \in S$, we have

$$shs^{-1} = s t_1^{u_1} t_2^{u_2} \cdots t_n^{u_n} s^{-1} = s t_1^{u_1} (s^{-1}s) t_2^{u_2} (s^{-1}s) \cdots (s^{-1}s) t_n^{u_n} s^{-1}$$

$$= \underbrace{(s t_1^{u_1} s^{-1})}_{\in H} \underbrace{(s t_2^{u_2} s^{-1})}_{\in H} \cdots \underbrace{(s t_n^{u_n} s^{-1})}_{\in H} \in H,$$

and $$s^{-1}hs = s^{-1} t_1^{u_1} t_2^{u_2} \cdots t_n^{u_n} s = s^{-1} t_1^{u_1} (ss^{-1}) t_2^{u_2} (ss^{-1}) \cdots (ss^{-1}) t_n^{u_n} s$$

$$= \underbrace{(s^{-1} t_1^{u_1} s)}_{\in H} \underbrace{(s^{-1} t_2^{u_2} s)}_{\in H} \cdots \underbrace{(s^{-1} t_n^{u_n} s)}_{\in H} \in H.$$

Step 2: Show that, $\forall g \in G$, $h \in H$ $\quad ghg^{-1} \in H$. ✓

$\forall g \in G$, $\exists s_1, \dots, s_n \in S$, $u_1, \dots, u_n \in \{\pm 1\}$ s.t. $\quad g = s_1^{u_1} s_2^{u_2} \cdots s_n^{u_n}$.

Then, $\forall h \in H$, $ghg^{-1} = (s_1^{u_1} s_2^{u_2} \cdots s_n^{u_n})\, h\, (s_1^{u_1} s_2^{u_2} \cdots s_n^{u_n})^{-1}$

$$= s_1^{u_1} s_2^{u_2} \cdots s_{n-1}^{u_{n-1}} s_n^{u_n} h\, s_n^{-u_n} s_{n-1}^{-u_{n-1}} \cdots s_2^{-u_2} s_1^{-u_1}$$

$$= s_1^{u_1} \underbrace{\left(s_2^{u_2} \cdots \underbrace{\left(s_{n-1}^{u_{n-1}} \underbrace{\left(s_n^{u_n} h\, s_n^{-u_n}\right)}_{\in H} s_{n-1}^{-u_{n-1}}\right)}_{\in H} \cdots s_2^{-u_2}\right)}_{\in H} s_1^{-u_1} \in H.$$

Therefore $H \trianglelefteq G$. $\blacksquare$

Pf of ii): If $|G| < \infty$ the proof is essentially the same, except that we can use the fact that $\forall g \in G$, $\exists s_1, \dots, s_n \in S$ s.t. $g = s_1 \cdots s_n$. (similarly for H)

This follows from the fact that, if $|G| < \infty$ then, $\forall s \in S$, we have that $|s| < \infty$, and $s^{|s|} = e \Rightarrow s^{-1} = s^{|s|-1}$. (so, we don't need to explicitly allow negative powers to get inverses of elements of S) $\blacksquare$

Exs:

6a) $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} = \langle i, j \rangle$

$H = \langle i \rangle = \{\pm 1, \pm i\}$

$i \cdot i \cdot i^{-1} = i \in H$ ✓

$j \cdot i \cdot j^{-1} = (j \cdot i) \cdot (-j) = (-k)(-j) = k \cdot j = -i \in H$.

Conclusion: $H \trianglelefteq G$.

6b) Lattice of subgroups of Q_8:

$$\begin{tikzcd}[column sep=small]
 & Q_8 = \langle i, j\rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle i \rangle \arrow[dr, dash] & \langle j \rangle \arrow[d, dash] & \langle k \rangle \arrow[dl, dash] \\
 & \langle -1 \rangle \arrow[d, dash] & \\
 & \{1\} &
\end{tikzcd}$$

$\langle i \rangle, \langle j \rangle, \langle k \rangle \trianglelefteq Q_8$ (similar argument as above)

$\langle -1 \rangle \trianglelefteq Q_8$ $\left(i(-1)i^{-1} = -1, \quad j(-1)j^{-1} = -1\right)$

This is an example of a non-Abelian group in which every subgroup is normal.